Differential Geometry extra exercises: complex structures

2025

By identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, we can define a complex structure on $\mathbb{C}^n$ as follows

$$J_{\mathbb{C}^n}(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}, \qquad J_{\mathbb{C}^n}(\frac{\partial}{\partial y_i}) = -\frac{\partial}{\partial x_i} \tag{1}$$

Where $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ are coordinates on $\mathbb{R}^{2n}$.

Exercise 1: Suppose $U \subseteq \mathbb{C}^n$ is an open and $F : U \to \mathbb{C}^m$ is smooth. Prove that F is holomorphic if and only if $(dF)_p \circ J_{\mathbb{C}^n} = J_{\mathbb{C}^m} \circ (dF)_p$. Use this to show that for any two complex manifolds M, N endowed with the canonical complex structure J_M and J_N, a smooth map $F : M \to N$ is holomorphic if and only if $(dF)_p \circ J_M = J_N \circ (dF)_p$.